Thinking mathematically – making sense and solving problems

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A mathematical way of thinking is becoming ever more necessary in a world that is being shaped by the impact of new technologies and the societal changes that these are bringing to every community. Attitudes and ways of knowing that enable new or unfamiliar tasks to be dealt with are now as essential as the procedures that have always been used to handle familiar tasks easily and efficiently. Making sense of the mathematics used also needs to be a key component, replacing speed in solving exercises and the external reward of a teacher's mark or approval as measures of success.

Mathematics is often thought of in terms of content such as number, space and measurement; procedures for computing, constructing or measuring; or applications across a wide range of situations. Yet, it is more aptly described as a way of thinking which allows concepts, processes and their uses to be built up, problems to be explored and solved, conjectures to be made and examined, and complex ideas about the world to be communicated in precise, succinct ways. Indeed, an ability to think with and about mathematics has replaced the memorisation of set procedures and the solution of routine problems as the focus of mathematics learning at all levels. Understanding number concepts and operations and the ways they might be expressed will still be crucial in underpinning these ideas, but so too will an ability to determine whether results, predictions and the implications based on them seem reasonable. Thus, making sense of mathematics needs to be a central concern, replacing speed in solving exercises and the external reward of a teacher's mark or approval as measures of success.

Number Sense

To think mathematically, students need to make sense of the ways numbers are used in everyday life to make judgments, interpret data and communicate effectively. They need to be able to work with numbers comfortably and competently, understanding the full range of meanings for numbers and computation as well as relationships among them. Consequently, number sense is not something that can be taught in discrete lessons but rather evolves over time as students come to terms with the way numbers are used in formulating and solving problems. Such critical ways of thinking require:

- understanding relationships among numbers
- appreciating the relative size of numbers
- a capacity to calculate and estimate mentally
- fluent processes for larger numbers
- adaptive uses of calculators
- an inclination to use understanding and facility in flexible ways

The following problems highlights the importance of having a sound sense of numbers:

- How many times do you say 'one' when counting to 200?
- How many times would you write 1 if you wrote the numbers to 200?
- What if you considered 2, 3, 4, 5, 6, …

	A	B	C	D
The counting numbers are arranged in four columns, A, B, C, D	**1**	**2**	**3**	**4**
• Under which column letter will 101 appear?	**8**	**7**	**6**	**5**
• Where would 1001, 10 001 appear?	**9**	**10**	**11**	**12**
• What if there were 5, 6, 7, 8, … columns?		**…**	**14**	**13**

Spatial Sense

Similarly, visual comprehension and understanding underpin many of the activities that people carry out in their daily lives, from making sense of maps and plans for buildings to assembling furniture or technical systems. Information is frequently presented in visual formats that need to be interpreted and processed, while the use of diagrams is often essential in coming to terms with problems or developing conceptual understanding of fractions, ratio and proportion. Spatial Sense requires:

- a capacity to visualise shapes and their properties
- understanding relationships among shapes and their properties
- an ability to link 2 dimensional and 3 dimensional representations
- basis for the thinking involved with ratios, fraction ideas
- presentation and interpretation of data and probability
- an inclination to use diagrams to visualize problem situations and applications

The following problems are examples of ways of using and developing spatial sense:

A farmer wants to make a new rectangular paddock that will be 26 m longer than it is wide. If she has 980 m of fencing, how long and how wide should she make the paddock?

- Peter spent 3 fifths of his money and had $120.00 left. How much money did he originally have?
- Peta spent 1 third of her money and then lost half of what she had left. She then had only $20.00. How much money did she have to start with?
- What is my number when: 3 ninths of my number is 35?
  3 eighths of my number is 48?
  63 is added to 1 fifth of my number, the result is double my number?

Solving Problems

Teaching problem solving has frequently centred on the use of particular strategies that could apply to various classes of problems (Schoenfeld 1992). However, many students are unable to access and use these strategies to solve problems outside of the teaching situation in which they have been introduced (Bond 1996). Rather than acquire a process for solving problems, these students attempt to memorise a set of procedures in the way that they have learned to cope with other aspects of the curriculum. Their view of mathematics focuses on a set of learned rules; success will follow the application of the right procedure to the appropriate numbers.

With no ability to judge what might be appropriate, the very approaches that have made them appear successful in routine mathematics have inhibited their ability to use and apply the mathematics they have 'mastered'. Further, their success in solving computational exercises and relatively simple one-step problems has led these students to believe that mathematical problems are simply a different vehicle for putting forward numbers that can be subjected to various computational procedures (Artzt and Armour-Thomas 1990, Bond 1996, Schoenfeld 1992). While their teachers were introducing strategies, students focussed on getting answers quickly that were invariably given simply as a bare number rather than an answer in terms of the problem itself. Such students are best described as *answer-focussed* . Yet, as Schoenfeld (2001: 53) later reminded us, "getting the right answer is only the beginning rather than the end ... an ability to communicate thinking convincingly is equally important".

In contrast, observation of successful problem solvers has shown that their success depends more on an analysis of the problem itself - what the question requires, what information might be used, what answer might be likely and so on - than having access to a variety of strategies (Schoenfeld 1994, Booker & Bond 1999, Fernandez 1994). A particular strategy might be used only after the intent of the problem was determined (Bond 1996). Thus, a first task in developing problem solving often requires deliberate steps to break down answer-focussed behaviour (Booker 2005: Workshop presented at the Mathematics Education into the 21st Century conference, Malaysia).

A problem solving process

Collaborative rather than individual work to develop problem solving skills is essential so that learners have opportunities to discuss solution methods rather than focus on 'the answer'. Determination of the meaning of a problem is essential before any plan is drawn up or work on a solution begins. Above all, the value and construction of an overall plan to manage the problem solving process is critical. However, many frameworks for managing problem solving are based on a linear model such as that of George Polya (1990). These do not cater for the processes of self-monitoring, self-regulation or self-assessment (Fernandez, 1994), leading to a view of problem solving as a procedure, where each step is completed independently with little thought given to the overall processes. Examining a problem and working through the process of solving it will reveal the thinking required for a productive problem solving model:

> Larry earns \$120 each week. On Thursday, he took some money to do his weekly shopping at the market and spent 3 fifths on fruit and vegetables and 1 third of what he had left at the butcher. Afterwards he met his friend Terry and bought them coffee for \$3.50 each. If he had \$5.00 left, how much did he take shopping?

Reading the problem and coming to terms with what the problem is asking necessitates delving beneath the surface level of the problem to *analyse* its structure. Larry took some money to do his shopping, spent some at the produce stall, some at the butcher shop and then had \$5 left. All of the information needed to solve this problem is available and no additional information is required. The final sentence asks how much money he started with but the real question is how much did he spend at the produce stall and butcher. To put it simply the final situation is known, something happened before this, and the initial situation needs to be determined.

Having *analysed* the problem and come to terms with its structure, possible ways in which it might be solved can now be *explore*d. Looking below the surface level and understanding what the problem is asking suggests ways of solving it. Possible ways that may come to mind during the analysis are:

materials – use counters to represent each dollar spent and work backwards through the problem from when he had \$5 left

guess and check - a series of guesses as to how much he took shopping could be made, checked and adjusted as needed

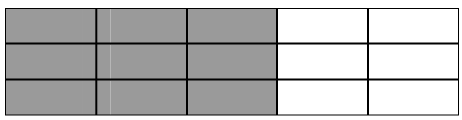
work backwards – he had \$5 at the end after shopping to the market

think of a similar problem – think about the problem and if it is similar to any other problem, if so think about how it was solved

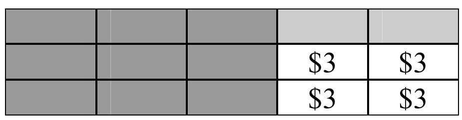
use a diagram – show what he spent in pictorial form to show each fraction spent and how much was left over

When the problem has been analysed and possible strategies or combination of means to a solution considered, a decision to *try* a particular way or combination of ways can be made. It should be borne in mind that there is no single best way to solve a problem but rather a strategy or combination of strategies would be used that makes most sense to the person solving the problem. The one shown here is based on the use of a diagram, an approach likely to shed light on many related problems:

Using a rectangle to represent the total amount Larry took shopping we can show by shading how much he spent each time at the produce stall and the butcher quite clearly. He spent $\frac{3}{5}$ of his money at the produce stall which can be shown by dividing the rectangle into five equal parts and shading 3 of them.



He then spent $\frac{1}{3}$ of the money he had left at the butcher. This can be shown by dividing the unshaded sections into 3 equal parts and shading one of these parts:



At this point he spent \$7 on coffee and had \$5 left. Therefore the four unshaded parts must be worth \$12 or \$3 per part. Since there are 15 parts, Larry must have started with \$45. All that remains is to go back to the problem and look at the solution in light of the problem and see if this answer is reasonable or not.

Discussion

This problem has served to identify the key elements within the problem solving process. To commence it was necessary to *analyse* the problem to unfold its meanings and discover its structure. Next to *explore* possible ways to solve the problem before selecting one to *try*. Having tried an idea, the answer(s) and solution needed to be *analysed* in light of the problem in case another solution or answer might be needed. As such, it shows the cyclical nature of the process and demonstrates to the learners how they have control over every step of the way from the initial reading to deciding that the problem has been dealt with.

It also provides a capacity to describe every step taken whenever students have a problem to solve or new mathematical content with which to come to terms. By discussing how they analysed the problem initially; explored various ways that might provide a solution; and then tried one or more possible solution paths that they were then able to analyse for completeness and sense making, they are not only applying metacognitive processes to their mathematical thinking but are also reinforcing the very methods that will give them success on future problems. This process brings to the fore the importance of understanding the problem and its structure before proceeding and has led to a model of the problem solving process (Booker et al, 2004):

- Read carefully
- What is the question asking?
- What is the meaning of the information?
  is it all needed? is there too little? too much?
- Which operations will be needed? what order?
- What sort of answer is likely?
- Have I seen a problem like this before?

Analyse
the problem

Plan to manage
Problem solving

Try
a solution strategy

- Use materials or a model
- Use a calculator
- Use pencil and paper
- Look for a pattern

Explore
means to a solution

- Use a diagram or drawing
- Work backwards
- Put information in a table
- Guess and check

Put the answer back into the problem to see if it makes sense

This model has proved to be invaluable at many levels (Bond 1996, Booker & Bond 1993, 1999). Firstly, it provides students with a process to follow in coming to terms with problems, determining solutions, and evaluating what they have done. At another level, it has provided them with a framework to discuss their solutions and value ways of thinking over the mere provision of answers. It has consequently fostered discussion among students as to a variety of solution paths, which ones might be personally preferred and how what seems a reasonable way when a problem is first solved might give way to a more elegant way when the fact that the problem is solvable is known. It has also proven to be a very effective way of changing students from trying to find 'the answer', to looking for other answers and justifying why the ones they have are all that are possible. The talking, listening, reading and writing about mathematics that this model promotes help students clarify their ideas and enable teachers to gain valuable insights into their students' thinking.

Thinking mathematically

A focus on problem solving as the central concern of mathematics education will require children to make sense of, and communicate in, the quantitative world they are inheriting (Steen 2001). This will allow them to read, write, experience, explain, discuss, defend, and clarify for themselves and others real mathematical issues (Elliot & Kenney 1996). In order to develop these problem solving attitudes and competencies, mathematical programs need to change from being strategy driven to focus on the understanding of what a problem actually requires and be based on an overall process that builds on the problem analysis. Understanding problem solving in this way will put thinking mathematically at the centre of learning and so empower students to increase their capacity to solve complex problems and develop new concepts and processes. It will also help them to build a capacity to persevere on difficult tasks and thus gain confidence in their own ways of knowing and acting. As the NCTM Principals and Standards for School Mathematics (2000) observes:

> *by learning problem solving in mathematics, student should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. In everyday life and in the workplace, being a good problem solver can lead to great advantages.*

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